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STATISTICAL SEISMIC RESPONSE ANALYSIS AND RELIABILITY DESIGN OF NONLINEAR STRUCTURE SYSTEM

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Industrial structure systems may have non-linearity, and are also sometimes exposed to the danger of earthquake. In the design of such system, these factors should be accounted for from the viewpoint of reliability. This paper proposes a method to analyze seismic response and reliability design of a complex non-linear structure system under random excitation. The actual random excitation is represented to the corresponding Gaussian process for the statistical analysis. Then, the non-linear system is subjected to this random process. The non-linear structure system is modelled by substructure synthesis method (SSM) procedure. The non-linear equations are expanded sequentially. Then, the perturbed equations are solved in probabilistic method. Several statistical properties of a random process that are of interest in random vibration applications are reviewed in accordance with the non-linear stochastic problem. The system performance condition in the design of system is that responses caused by random excitation be limited within safe bounds. Thus, the reliability of the system is considered according to the crossing theory. Comparing with the results of the numerical simulation proved the efficiency of the proposed method.

1. INTRODUCTION

In recent years, the trend in mechanical systems has been toward high speed and lightweight ones in many industrial machines. These conditions can cause trouble of a non-linear vibration in mechanical systems. Hence, it has become important to consider the non-linear characteristics in vibration analysis, design of the structure system. Iwatsubo *et al.* [1, 2] have proposed a new method to analyze the vibration of a multi-degrees-of-freedom (m-d.o.f.) non-linear mechanical system. Moon *et al.* [3, 4] have reported study on the vibration of mechanical system to analyze the dynamic problems of non-linear m-d.o.f. systems. They developed the SSM technique to reduce the overall size of the problem for the non-linear structure, and obtained approximate solutions of the non-linear system using a perturbation method.

On the other hand, it is necessary that a high-speed system used for the jet engine of an aircraft, power plant turbine, etc. promptly pass a critical speed. Accordingly, the casing is often modelled elastically to decrease the critical speed. When random process excites such a mechanical system, it is possible that the casing is excited to contact with the bearing and there is a danger that the bearing will be damaged. Therefore, the investigation of the random response of rotating machinery is very important from the viewpoint of disaster protection.

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Soni *et al.* [5] and Srinivasan *et al.* [6] have reported the earthquake analysis of rotor system using the response spectrum method and time response method in deterministic system. Matsushita *et al.* [7] and Azuma *et al.* [8] analyzed the seismic response of the rotor system using the modal analysis method with real earthquake data. They used the real earthquake data to analyze the linear response. From the viewpoint of the dynamic response of mechanical system against random excitation, it can be treated as stochastic problem. However, an approach to the vibration of non-linear rotor systems under seismic waves has not yet been tried. Moreover, the reliability analysis of a non-linear rotor-bearing-casing system utilizing a statistical approach to a seismic wave is not found in past research.

Therefore, this paper proposes an analytical method for non-linear vibration and reliability of mechanical system against a random excitation by applying the statistical method. This paper deals the reliability of a non-linear mechanical system under the actual random excitation, while regarding earthquake excitation as a stationary random process. The possibility of failure is obtained by assuming that a failure of the system occurs when the response crosses over the safe bounds. Then, several statistical properties that are of interest in non-linear random vibration applications are reviewed.

2. METHOD OF ANALYSIS

The mechanical structure system with the non-linear restoring force of the system, which is excited by earthquake, is considered as shown in Figure 1. The excitation is regarded as a random process; hence, it is extremely difficult to obtain an exact solution. Thus, solutions can be obtained approximately. It should be noted that the solution itself for random inputs is not the ultimate goal in stochastic analysis of a non-linear system; instead, more relevant information is the statistical properties of the amplitude of the response. To elaborate, a non-linear system with the inputs, which are assumed to be a Gaussian random process, is considered. Because of the non-linear characteristic, the output is no longer a Gaussian random process; hence, the statistic characteristic of its amplitude cannot be evaluated through the PDF (probability density function).

Therefore, an adequate method to evaluate the statistical properties of the response of a non-linear structure system should be developed. For this reason, the random excitation is approximated to Gaussian stationary process by reasonable procedures. Then, the non-linear equation of motion is reinstated with approximated Gaussian process. After that,

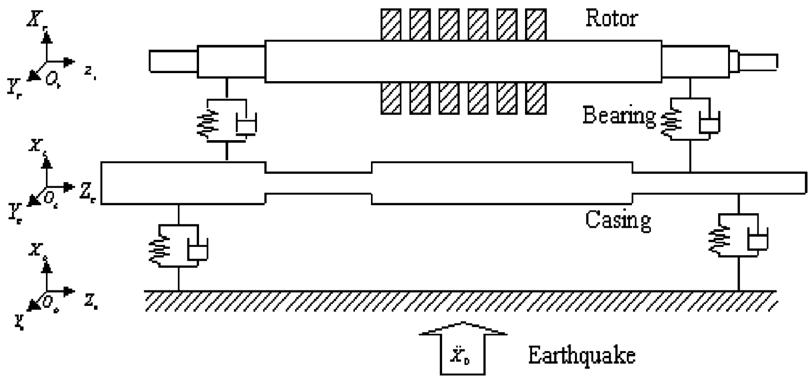

Figure 1. Mechanical system for analysis.

the perturbation theory is applied to solve non-linear equation of motion. Finally, the statistics properties of nonlinear response are obtained.

2.1. NON-LINEAR SYSTEM EXCITED BY RANDOM PROCESS

For the simple explanation, equation of motion of the arbitrary mode of non-linear system can be expressed as

$$\ddot{x} + 2\varsigma\omega_0\dot{x} + \omega_0^2 x + \varepsilon\omega_0^2 x^3 = \ddot{X}_0, \tag{1}$$

where ς, ω_0 are damping ratio and natural frequency respectively. ε is a small parameter. Equation (1) has a non-linear restoring force, which can be expressed in higher order terms of the displacement, $\varepsilon\omega_0^2 x^3$. $\ddot{X}_0$ is random excitation, which has a spectrum density $S_N(\Omega)$. Meaning of the stochastic analysis such as equation (1) is to decide the statistic information of displacement x. Generally, the statistic characteristic of random process is decided from the PDF and power spectrum density (PSD) function of the system. Accordingly, for the probabilistic analysis of non-linear random response, PSD and PDF of excitation forces should be obtained.

It is clear that ground acceleration is inherently non-stationary, treating a typical record of earthquake induced ground acceleration [9]. However, if the principal shock duration is limited to the period corresponding to the strong-motion portion over which the peak structural response occurs, a stationary process appears to be a good approximation. Hence, this study deals with the response of a non-linear system under ground excitation of the stationary random process by considering the strong motion duration of the earthquake.

Figure 2 shows the random excitation of the system and its PSD and PDF, which is regarded as narrow band process. Important values of statistic properties are mean (= −0·072) and peak frequency (= 18·75 rad/s) and maximum value of acceleration (175·89 gal). Figure 2(c) shows the corresponding erratic PSD and the fitted PSD function, which shows relatively good agreement. The computed PSD is obtained from the part of earthquake data during 3–18 s, which can be regarded as stationary process from the viewpoint of the amplitude envelope of earthquake graph, as shown in Figure 2(b). The functional form for the spectral density function of earthquake motion is

$$S_N(\Omega) = \frac{\omega_g^2\alpha^2\Omega^2}{\left(\omega_g^2 - \Omega^2\right)^2 + 4\varsigma_g^2\omega_g^2\Omega^2} S_0^2, \tag{2}$$

where ω_g, ς_g and S_0 are a dominant frequency, damping ration of filter and spectrum intensity of random process respectively. α is a maximum value of input. For instance, Taft earthquake (1952) has values of $\varsigma_g = 0{\cdot}41$, $\omega_g = 18{\cdot}75\,\mathrm{rad/s}$, $\alpha = 1{\cdot}75\,\mathrm{m/s^2}$ and $S_0 = 0{\cdot}0132\,\mathrm{m^2/sec^3}$ respectively. From Figure 2(c), it is estimated that $S_N(\Omega)$ can be applicable to solve the response statistically.

PDF of input is obtained from the part of earthquake data during 0–50 s, during 3–18 s and during 5–13 s, as shown in Figure 2(d). PDF which is obtained from the part of earthquake data during 3–18 s shows a reasonable Gaussian distribution. The part of the earthquake during 3–18 s has the mean (= −0·0062). Thereby, the strong part of earthquake excitation process can be regarded as Gaussian stationary random process with mean zero. The PDF can be expressed as

$$P(\ddot{X}_0) = \frac{1}{\sqrt{2\pi\sigma_{\ddot{X}_0}}} \exp\left(-\frac{\ddot{X}_0^2}{2\sigma_{\ddot{X}_0}^2}\right), \tag{3}$$

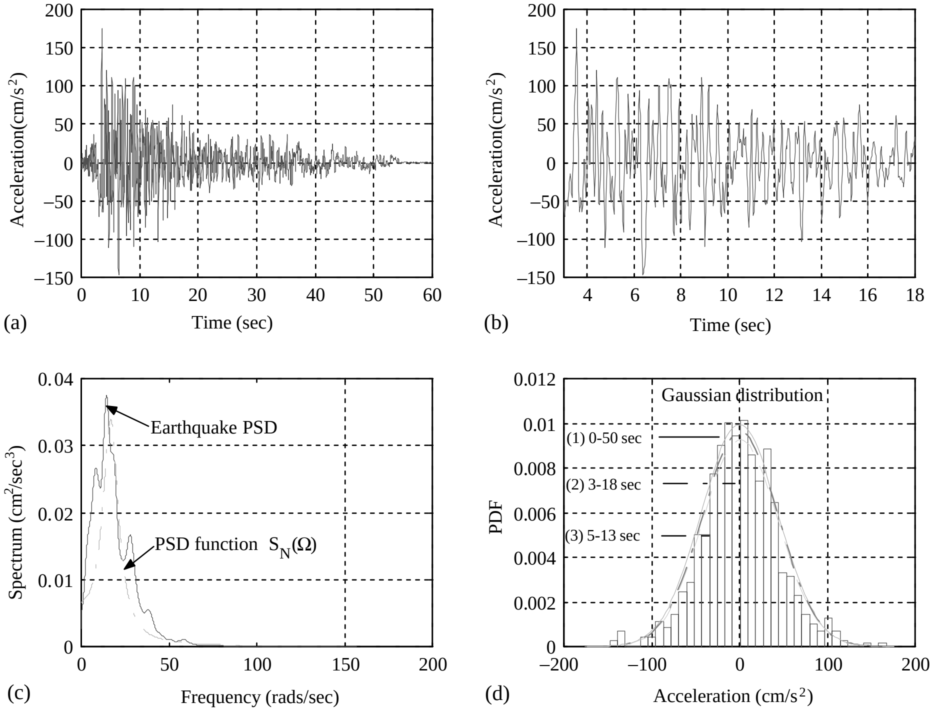

Figure 2. Random excitation force and its PSD, PDF. (a) Taft earthquake (1952, S69E), (b) strong part of earthquake (c) PSD function, (d) PSD function.

where $\sigma^2_{\ddot{X}_0}$, $\sigma_{\ddot{X}_0}$ are the variance of excitation and its root mean square. Then, non-linear equation of motion can be reinstated as

$$\ddot{x} + 2\varsigma\omega_0\dot{x} + \omega_0^2 x + \varepsilon\omega_0^2 x^3 = W(t), \tag{4}$$

where $W(t)$ is a Gaussian stationary narrow band random process with zero-mean. $x(t)$ is the stationary response of a linearly damped Duffing system subjected to a Gaussian process excitation.

Here, $x(t)$ is not Gaussian process, generally. However, if the system is lightly damped and if the non-linearity is small (that is, $\varepsilon \leqslant 1$), then $x(t)$ is still expected to be a narrow band random process, as shown in Figure 3, numerical example. The response has the mean ($=-0{\cdot}0044$). PDF of non-linear response shows a Gaussian distribution. Thus, the response of Equation (4) can be analyzed statistically by applying the perturbation theory. In Equation (4), x can be perturbed as $x = x_0 + \varepsilon x_1$. This replaces the solution of the non-linear random vibration problem represented by equation (4) with the solution of a series of linear random vibration problems with the same differential form but different inputs

$$\ddot{x}_0 + 2\varsigma\omega_0\dot{x}_0 + \omega_0^2 x_0 = W(t), \tag{5}$$

$$\ddot{x}_1 + 2\varsigma\omega_0\dot{x}_1 + \omega_0^2 x_1 = -\omega_0^2 x_0^3 = f_p(x_0), \tag{6}$$

where $f_p(x_0) = \omega_0^2 x_0^3$. From equations (5) and (6), the approximating functions $x(t)$ can be obtained sequentially. Here, zeroth order approximation x_0 is Gaussian process. Its probabilistic characteristics can be obtained by classical methods of linear random vibration. However, the characterization of x_1 is less simple because the input is now a

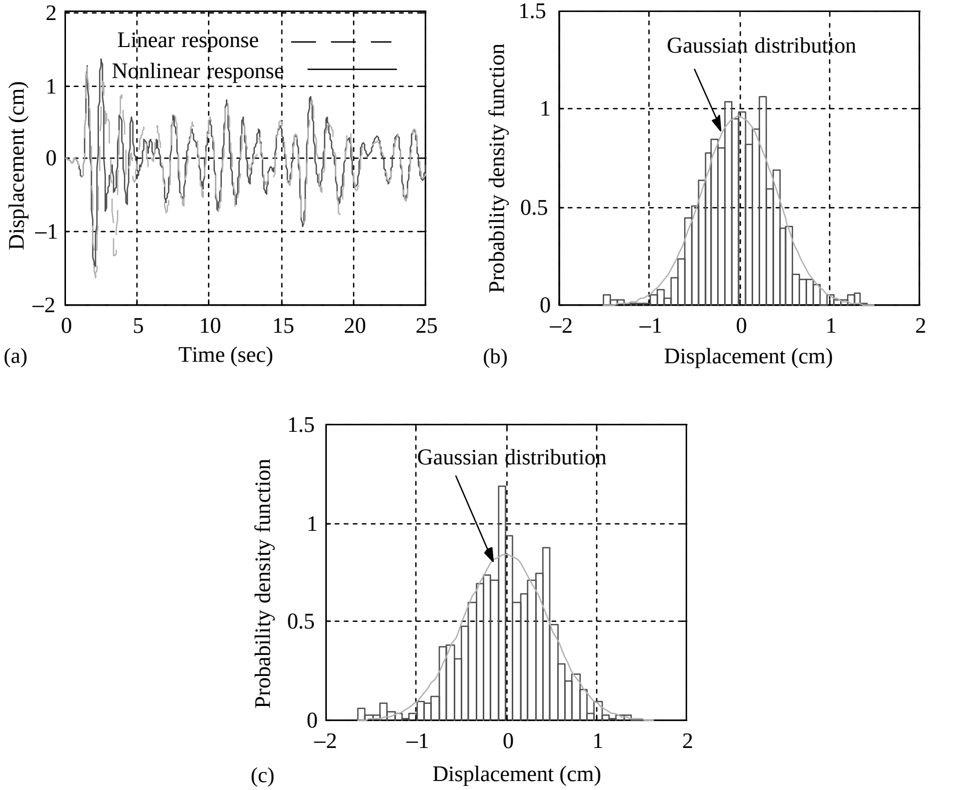

Figure 3. Non-linear response x, linear response x_0 and their PDF ($\varepsilon = 0{\cdot}3$, $\varsigma = 0{\cdot}1$, $\omega = 5{\cdot}23$). (a) Time response, (b) PDF of non-linear response, (c) PDF of linear response.

non-Gaussian process whose mean and covariance function are not generally available in a closed form. Moreover, the determination of the second moment characterization of the approximate solution $x = x_0 + \varepsilon x_1$ also requires the correlation function of x_0, x_1, which is not readily available. In handling this problem, the objective is the determination of the stationary mean and covariance function of the first order approximation, as given in equation (6). Covariance function becomes due to its stationary characteristic

$$E\{x(t+\tau)x(t)\} = R(\tau) = E\{x_0(t+\tau)x_0(t)\} + \varepsilon E\{x_0(t+\tau)x_1(t)\} + \varepsilon E\{x_1(t+\tau)x_0(t)\}, \quad (7)$$

where each term can be obtained from the random vibration of linear system as

$$E\{x_0(t+\tau)x_0(t)\} = \int_0^\infty \int_0^\infty E\{W(t+\tau-\theta_1)W(t-\theta_2)\}h(\theta_1)h(\theta_2)\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2, \quad (8)$$

$$E\{x_0(t+\tau)x_1(t)\} = -\int_0^\infty \int_0^\infty E\{W(t+\tau-\theta_1)f_p(t-\theta_2)\}h(\theta_1)h(\theta_2)\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2, \quad (9)$$

$$E\{x_1(t+\tau)x_0(t)\} = -\int_0^\infty \int_0^\infty E\{W(t-\theta_1)f_p(t+\tau-\theta_2)\}h(\theta_1)h(\theta_2)\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2, \quad (10)$$

where $h(\cdot)$ is the impulse response function corresponding to $\varepsilon = 0$. The determination of the expectations in equations (8)–(10) involve lengthy but straightforward calculations of expectations of polynomials in Gaussian variables. Thus the covariance function of the non-linear response can be evaluated. Then, the spectral density of the non-linear response is obtained by taking the Fourier transform of equation (7).

$$S_x(\Omega) = S_N(\Omega)|H(\Omega)|^2\Big[1 - 6\varepsilon\omega_0^2\sigma_{x_0}^2\,\mathrm{Re}\{H(\Omega)\}\Big], \tag{11}$$

where $\sigma_{x_0}^2$ is the stationary variance of the linear response. $H(\Omega)$ is frequency response function of linear equation between the excitation and the displacement of response. The corresponding variance can be obtained from the covariance $R(\tau)$ of the system by letting $\tau = 0$, which is the same value of the mean-square response of the non-linear vibration. Since the mean response, $E[x(t)]$, is zero, the variance is equal to the second moment $E[x(t)]$

$$E[x^2(t)] = R(0) = \sigma_{x_0}^2\left[1 - 6\varepsilon\omega_0^2\int_0^\infty \{R(\tau)h(\tau)\}\,\mathrm{d}\tau\right]. \tag{12}$$

The mean-square value of the linear response in terms of the system response function and the spectral density of the input is

$$\sigma_{x_0}^2 = \int_0^\infty S_N(\Omega)|H(\Omega)|^2\,\mathrm{d}\Omega. \tag{13}$$

This procedure is applied to analyze a complex M-d.o.f. non-linear system.

2.2. RELIABILITY ANALYSIS BY CROSSING THEORY

Here, reliability analysis through the crossing theory is considered. Unsatisfactory performance is assumed to be caused by excessive deformation by a gradual accumulation of damage under cyclic loading. The reliability of structural systems depends primarily on crossing characteristics of response process representing deformations. The crossings of response correspond to crossings of level X of response with positive slope or X-upcrossings of $x(t)$. The mean X-upcrossing rate of $x(t)$ is

$$\upsilon_X^+(t) = E\{\dot{x}(t) + |x(t) = X\}p(X,\, t), \tag{14}$$

where $p(X, t)$, denotes the PDF of $\{x(t), \dot{x}(t)\}$, which can be obtained using techniques for analyzing the response of linear. Note that $\upsilon_X^+(t) = \upsilon_X^+$ is time-invariant for stationary response. The mean X-upcrossing rate of the non-linear displacement $x = x_0 + \varepsilon x_1$ can be obtained as

$$\upsilon_X^+(t) = \frac{\omega_0\sigma_{x0}\sqrt{\dfrac{\varepsilon}{\pi}}}{K_{1/4}\left(\dfrac{1}{8\varepsilon\sigma_{x0}^2}\right)}\exp\left(-\frac{1}{8\varepsilon\sigma_{x0}^2}\right)\exp\left(-\frac{1}{2\sigma_{x0}^2}\left(x^2 + \frac{\varepsilon}{2}x^4\right)\right), \tag{15}$$

where $K_{1/4}$ is the modified Bessel function of order $\frac{1}{4}$.

2.3. MODELLING OF A COMPLEX NON-LINEAR STRUCTURE

For the analysis, the system shown in Figure 1 is considered as a mechanical system. The rotor has non-linearity with respect to its material property. For the dynamic analysis of complex systems, the SSM can be applied. The overall system is divided into three components, i.e., the rotor is the non-linear component, the casing is the linear component, and the bearing is the assembling component. The acceleration of gravity is ignored for simplicity of analysis. The rotor and casing components are modeled using finite element method (FEM).

The co-ordinates of the rotor-bearing-casing system are shown in Figure 1. The O_r-$X_rY_rZ_r$ co-ordinate system is fixed in rotor, such that the origin coincides with the center of the shaft where the X_r-axis is vertically upwards, the Y_r-axis is horizontal and perpendicular to the shaft, and the Z_r-axis is along shaft. The O_c-$X_cY_cZ_c$ co-ordinate system is fixed in casing. The O_0-$X_0Y_0Z_0$ is an absolute co-ordinate system, which is fixed in basement. The $U_r(=X_r-X_c,\ Y_r-Y_c,\ Z_r-Z_c)$ is a relative displacement between rotor and casing. The $U_c(=X_c-X_0,\ Y_c-Y_0,\ Z_c-Z_0\ )$ is a relative displacement between casing and basement. $\ddot{X}_0$ is an acceleration of the earthquake input.

2.4. MODELLING OF NON-LINEAR COMPONENT

In the current rotating machinery, the non-linear vibration phenomena sometimes occur in the shrinkage fit rotor, assembly rotor, power plant rotor with coil, and in high polymer rotor. These phenomena can be modelled with a non-linear restoring force, which can be expressed in higher order terms of the displacement. To apply the SSM to those complex system, equation of motion is obtained for the non-linear component [1–4]. The external force is considered as the unbalance force and the earthquake. Internal force is considered because the non-linear component can be synthesized through the internal force with the other components.

$$\begin{aligned}&[{}^1M]\{{}^1\ddot{U}_r\}+[{}^1K]\{{}^1U_r\}+\varepsilon[K_N]\{{}^1U_r^3\}\\&\quad=\{{}^1F_u(t)\}+\{{}^1F_E\}+\{{}^1F_b\},\end{aligned}\tag{16}$$

where $[{}^1M]$, $[{}^1K]$ and $[K_N]$ are a mass matrix, stiffness matrix and non-linear stiffness term respectively. $\{{}^1F_u(t)\},\{{}^1F_b\}$ are an unbalance excitation of rotor and an internal force vector. The earthquake force is

$$\{{}^1F_E\}=-[{}^1M]\{I\}\{\ddot{X}_0\},\tag{17}$$

where $\{I\}$ is a vector which shows the direction. In order to apply modal analysis, modal co-ordinate system $\{{}^1\xi\}$ is introduced by using the modal matrix $[{}^1\Phi]$ of the linear system. Then, the displacement $\{{}^1U_r\}$ is transformed into the modal co-ordinate approximately as [3, 4]

$$\begin{aligned}&\{{}^1\ddot{\xi}\}+[\backslash{}^1\omega^2_\backslash]\{{}^1\xi\}+\varepsilon[\backslash{}^1k_{N\backslash}]\{{}^1\xi^3\}\\&\quad=\{{}^1f_u(t)\}+\{{}^1f_E\}+\{{}^1f_b\}\end{aligned}\tag{18}$$

where $\{{}^1f_u(t)\},\{{}^1f_b(t)\}$ and $\{{}^1f_E\}$ are an unbalance force, an internal force and an external force against earthquake in modal co-ordinates respectively. $[\backslash{}^1\omega^2_\backslash],\varepsilon[\backslash{}^1k_{N\backslash}]$ are an eigenvalue matrix and a non-linear term in modal co-ordinates.

According to the previous section, the earthquake can be expressed with approximated Gaussian stationary random process $W(t)$. For the simple expression of external force equation to describe the SSM method, those external forces are expressed in a term as $\{{}^1W(t)\}\approx\{{}^1f_u(t)\}+\{{}^1f_E\}$. Thus equation (18) can be expressed in a compact form as

$$\{{}^1\ddot{\xi}\}+[\backslash{}^1\omega^2_\backslash]\{{}^1\xi\}+\varepsilon[\backslash{}^1k_{N\backslash}]\{{}^1\xi^3\}=\{{}^1W(t)\}+\{{}^1f_b\}.\tag{19}$$

Here, the perturbation method is applied to solve the non-linear equation. The small variant ε can be regarded as the perturbation parameter, because the variant $\varepsilon[\backslash{}^1k_{N\backslash}]$ is small relative to $[{}^{\backslash 1}\omega^2_{\backslash}]$. $\{{}^1\xi\}$ can be expanded in terms of a series of ε

$$\{{}^1\xi\} = \{{}^1\xi^{(0)}\} + \varepsilon\{{}^1\xi^{(1)}\} + \dots, \tag{20}$$

where superscripts (0), (1) denote the perturbation order. Then, the perturbed equations are evaluated as

$$\begin{aligned} \{{}^1\ddot{\xi}^{(0)}\} + [{}^{\backslash 1}\omega^2_{\backslash}]\{{}^1\xi^{(0)}\} &= \{{}^1W^{(0)}\} + \{{}^1f_b^{(0)}\}, \\ \{{}^1\ddot{\xi}^{(1)}\} + [{}^{\backslash 1}\omega^2_{\backslash}]\{{}^1\xi^{(1)}\} &= \{{}^1f_p({}^1\xi^{(0)})\} + \{{}^1f_b^{(1)}\}, \end{aligned} \tag{21}$$

where $\{{}^1f_p({}^1\xi^{(0)})\} = -[{}^{\backslash 1}k_{N\backslash}]\{{}^1\xi^{(0)3}\}\cdot\{{}^1W^{(0)}\}$ is external force term, which is expressed in perturbation zeroth order. $\{{}^1f_b^{(0)}\}$ and $\{{}^1f_b^{(1)}\}$ are perturbed internal forces, which are obtained from the relation of $\{{}^1f_b\} = \{{}^1f_b^{(0)}\} + \varepsilon\{{}^1f_b^{(1)}\} + \cdots$.

2.5. EQUATIONS OF MOTION OF AN ASSEMBLED SYSTEM

To apply the SSM, the casing of rotor system is modelled as linear component and the equation of motion is obtained readily. After the eigenvalue analysis, the displacement can be transformed into modal co-ordinate as $\{{}^2U_c\} \equiv [{}^2\Phi]\{{}^2\xi\}$. $[{}^2\Phi]$, $\{{}^2\xi\}$ are a modal matrix and modal co-ordinate of casing. Even the casing is linear system, this component is perturbed as same as the non-linear component, because the higher order harmonic oscillation which is occurred in the non-linear component is translated through the higher order perturbed equation as [4]

$$\begin{aligned} \{{}^2\ddot{\xi}^{(0)}\} + [{}^{\backslash 2}\omega^2_{\backslash}]\{{}^2\xi^{(0)}\} &= \{{}^2W^{(0)}\} + \{{}^2f_b^{(0)}\}, \\ \{{}^2\ddot{\xi}^{(1)}\} + [{}^{\backslash 2}\omega^2_{\backslash}]\{{}^2\xi^{(1)}\} &= \{{}^2f_b^{(1)}\}, \end{aligned} \tag{22}$$

where $\{{}^2f_b^{(0)}\}, \{{}^2f_b^{(1)}\}$ are the perturbed internal forces.

$\{{}^2W^{(2)}\}$ is the external force term, which is expressed with approximated Gaussian stationary random process $W(t)$. Though the internal force, equation (22) can be assembled with equation (21). As an assembling component in SSM, simple linear ball bearings are considered. Generally, there is a damping term in the bearing, but it is ignored in this study. The restoring force of the bearing is modelled as linear term, where the force and displacement are expressed as

$$[{}^1k_{b1}]\{{}^1U_{rb}\} = \{{}^1f_b\}, \ -[{}^2k_{b2}]\{{}^2U_{rb}\} = \{{}^2f_b\}, \tag{23}$$

where $[{}^jk_{bj}](j = 1, 2)$ are bearing coefficients. $\{{}^1f_b\}$, $\{{}^2f_b\}$ are the internal force vectors of the non-linear component and linear component respectively. $\{{}^jU_{rb}\}$ is a relative displacement between the rotor and casing corresponding to the bearing. To synthesize each component through the assembling component, the order of equation is arranged. The perturbation parameter ε of the non-linear component is introduced. Then, the displacement can be expressed as

$$\{{}^jU_{rb}\} = \{{}^jU_{rb}^{(0)}\} + \varepsilon\{{}^jU_{rb}^{(1)}\}. \tag{24}$$

And the internal force vectors are perturbed as

$$\{{}^jf_b\} = \{{}^jf_b^{(0)}\} + \varepsilon\cdot\{{}^jf_b^{(1)}\}. \tag{25}$$

In SSM, each component is synthesized to entire system. In order to synthesize each component, equations (21), (22) and (25) are combined and rewritten according to the perturbation order $\varepsilon^{(l)}$ $(l = 0,\ 1)$.

$$\{\ddot{\xi}^{(l)}\} + [\bar{K}^{(l)}]\{\xi^{(l)}\} = \{F^{(l)}(t)\}, \tag{26}$$

$$\{\xi^{(l)}\} = \left\{\{^1\xi^{(l)}\}^{\mathrm{T}},\ \{^1U_{rb}^{(l)}\}^{\mathrm{T}},\ \{^2U_{rb}^{(l)}\}^{\mathrm{T}},\ \{^2\xi^{(l)}\}^{\mathrm{T}}\right\}^{\mathrm{T}},$$

$$\{F(t)^{(l)}\} = \left\{\{^1W^{(l)}\}^{\mathrm{T}},\ \{-^1f_b^{(l)}\}^{\mathrm{T}},\ \{^2f_b^{(l)}\}^{\mathrm{T}},\ \{^2W^{(l)}\}^{\mathrm{T}}\right\}^{\mathrm{T}}$$

and $[\bar{K}^{(l)}]$ is the stiffness matrix of the overall system, which is composed all of the component stiffness. According to the synthesizing procedure of SSM, the reduced order of degrees of freedom for overall system is obtained by modal truncation of each component. The equation of order $\varepsilon^{(l)}$ is obtained as

$$\begin{Bmatrix} ^1\ddot{\xi}_i^{(l)} \\ ^2\ddot{\xi}_i^{(l)} \end{Bmatrix} + \begin{bmatrix} [^1\omega_i^2] + [a_1] & [a_2] \\ [a_3] & [^2\omega_i^2] + [a_4] \end{bmatrix} \begin{Bmatrix} ^1\xi_i^{(l)} \\ ^2\xi_i^{(l)} \end{Bmatrix} = \begin{Bmatrix} ^1f_\eta^{(l)} \\ ^2f_\eta^{(l)} \end{Bmatrix}, \tag{27}$$

$$[a_1] = [\phi_{b1}]^{\mathrm{T}}[^1k_{b1}][\phi_{b1}], \quad [a_2] = [\phi_{b1}]^{\mathrm{T}}[^2k_{b1}][\phi_{b2}],$$

$$[a_3] = [\phi_{b2}]^{\mathrm{T}}[^1k_{b2}][\phi_{b1}], \quad [a_4] = [\phi_{b2}]^{\mathrm{T}}[^2k_{b2}][\phi_{b2}],$$

$$\left\{\{^1f_\eta^{(l)}\},\ \{^2f_\eta^{(l)}\}\right\}^{\mathrm{T}} = \left\{[\phi_{a1}]^{\mathrm{T}}\cdot\{^1W^{(0)}\}^{\mathrm{T}},\ [\phi_{a2}]^{\mathrm{T}}\cdot\{^2W^{(0)}\}^{\mathrm{T}}\right\}^{\mathrm{T}}$$

where, $[\phi_{aj}]$ is the eigenvector matrix of each component except the assembling region. $[\phi_{bj}]$ is the eigenvector matrix of the assembling region, which is derived from the eigenvector of each substructure corresponding to its bearing. The external force term of order $\varepsilon^{(1)}$ is obtained as

$$\begin{aligned} &\left\{\{^1f_\eta^{(1)}\}, \{^2f_\eta^{(1)}\}\right\}^{\mathrm{T}} \\ &= \left\{[\phi_{a1}]^{\mathrm{T}}\cdot\{-[^{\backslash 1}k_{N\backslash}]\{^1\xi^{(0)3}\}\} + [\phi_{b1}]^{\mathrm{T}}\cdot\{^1f_b^{(1)}\},\ [\phi_{a2}]^{\mathrm{T}}\cdot\{0\} + [\phi_{b2}]^{\mathrm{T}}\cdot\{^2f_b^{(1)}\}\right\}^{\mathrm{T}}. \end{aligned}$$

3. EVALUATION OF SYSTEM PERFORMANCE

In this section, the statistical properties of non-linear system vibration are obtained, and a reliability design is evaluated in a statistical sense.

3.1. STATISTICAL PROPERTIES OF NON-LINEAR RESPONSE

The response of non-linear random vibration is solved statistically in an overall system. The earthquake is used as the excitation wave, which is regarded as the Gaussian stationary random process, by considering the strong motion duration. When the statistical properties of an earthquake are known, statistical properties of the system response can be obtained.

After the eigenvalue analysis of the overall system with equation (27), the order $\varepsilon^{(l)}$ co-ordinate $\{\eta^{(l)}\}$ of the overall system is introduced for modal analysis as

$$\{\xi^{(0)}\} \equiv [\Phi_t]\{\eta^{(0)}\}, \quad \{\xi^{(1)}\} \equiv [\Phi_t]\{\eta^{(1)}\}. \tag{28}$$

where $[\Phi_t]$ is the eigenvector matrix of the overall system. The equation of motion of order $\varepsilon^{(0)}$ is

$$\ddot{\eta}_i^{(0)} + 2\zeta_i\omega_{ti}\dot{\eta}_i^{(0)} + \omega_{ti}^2\eta_i^{(0)} = W_{\eta i}^{(0)}, \quad (i = 1,\ 2,\ 3,\ldots,\ n), \tag{29}$$

where ω_{ti}^2 is the eigenvalue of the overall system. $W_{\eta i}^{(0)}$is the external force term in modal co-ordinates. The damping of the system is assumed to be the proportional damping of the eigenvalue. According to the linear random vibration theory, the solution $\eta_i^{(0)}(t)$ of the linear differential equation may be readily obtained. Then, the equation of motion of order $\varepsilon^{(1)}$ can be described as

$$\ddot{\eta}_i^{(1)} + 2\zeta\omega_{ti}\dot{\eta}_i^{(1)} + \omega_{ti}^2\eta_i^{(1)} = f_{\eta i}^{(1)}(\eta_i^{(0)}), \quad (i = 1, 2, \ldots, n), \tag{30}$$

where $f_{\eta i}^{(1)}(\eta_i^{(0)})(= -\beta_i^2\eta_i^{(0)3})$ is the external force term. β_i^2 is the non-linear coefficient. The response is

$$\eta_i^{(1)}(t) = -\beta_i^2\int_0^\infty \eta_i^{(0)3}(t-\tau)h_i(\tau)\,\mathrm{d}\tau, \tag{31}$$

where $h_i(t)$ is the impulse response function of the linear system. Accordingly, the response of a non-linear system can be evaluated as

$$\eta_i = \eta_i^{(0)} + \varepsilon\eta_i^{(1)}. \tag{32}$$

The equations of $\eta_i^{(0)}$, $\eta_i^{(1)}$ can be used to compute various statistical properties of the response. The covariance of the non-linear response, computed to the first order of ε, can be obtained as

$$R_{\eta i}(\tau) = \int_0^\infty \left\{\tfrac{1}{2}S_{Ni}^{(0)}(\Omega)|H_i(\Omega)|^2 - \tfrac{3}{2}\varepsilon\sigma_{\eta i}^{(0)2}S_{Ni}^{(0)}(\Omega)|H_i(\Omega)|^2H_i^*(\Omega)\cos\Omega\tau\right\}\mathrm{d}\Omega, \tag{33}$$

where $H_i^*(\Omega)$ is conjugate function of $H_i(\Omega)$. $S_{Ni}^{(0)}(\Omega)$ is the spectral density of the excitation. Then, the spectral density $S_{\eta i}(\Omega)$ of the non-linear response is obtained by taking the Fourier transform of the covariance function as

$$S_{\eta i}(S_{\eta i}(\Omega) = S_{Ni}^{(0)}(\Omega)|H_i(\Omega)|^2\left[1 - 6\varepsilon\beta_i^2\sigma_{\eta i}^{(0)2}\,\mathrm{Re}[H_i(\Omega)]\right], \tag{34}$$

where $\mathrm{Re}[H_i(\Omega)]$ is the real part of $H_i(\Omega)$. The corresponding variance can be obtained from the covariance $R_{\eta i}(\tau)$ of the system by letting $\tau = 0$, which is the same value of the mean-square response of the non-linear vibration

$$\sigma_{\eta i}^2 = \sigma_{\eta i}^{(0)2}\left[1 - 6\varepsilon\beta_i^2\int_0^\infty \{R_{\eta i}(\tau)h_i(\tau)\}\,\mathrm{d}\tau\right]. \tag{35}$$

The stationary variance $\sigma_{\eta i}^{(0)2}$ is the mean-square value of the linear response. Examining $\sigma_{\eta i}^2$, it appears that if the system is non-linear with light damping, weak non-linearity and the excitation random process is Gaussian stationary, then the response spectral density, covariance function, and mean-square value, can all be calculated from the knowledge of the spectral density of the excitation process and the magnitude of the frequency response $|H_i(\Omega)|$.

3.2. RELIABILITY ANALYSIS BY POSSIBILITY OF FAILURE

Generally, the evaluation of seismic response of rotating machinery is concerned with the maintenance of the operating ability against the seismic excitation. This can be verified by the possibility of failure by the contact between the bearing and casing. The possibility of failure is obtained by assuming that a failure of the system occurs when the response crosses over the safe bound, as shown in Figure 4. The displacement X_r is used as the standard point for the evaluation of the failure problem. The probability that X_r exceeds safe set B gives the probability of the system failure, which is constrained by casing to prevent the damage of bearing. B is the limited amplitude of the rotor. The mean B-

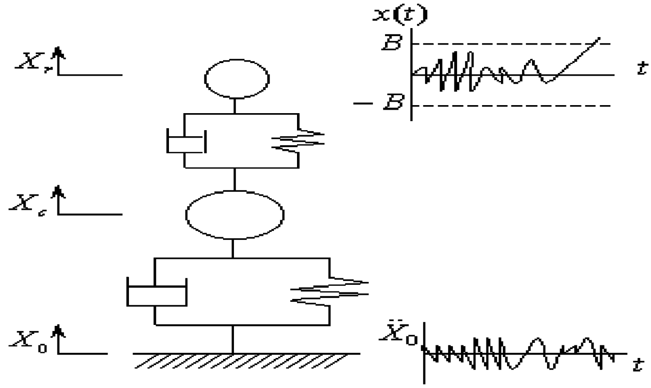

Figure 4. Possibility of failure problem.

upcrossing rate of the non-linear vibration of the system can be obtained as

$$\upsilon_B^+(t) = \frac{\omega_{t0}\sigma_{\eta_0}^{(0)}\sqrt{\dfrac{\varepsilon}{\pi}}}{K_{1/4}\left(\dfrac{1}{8\varepsilon\sigma_{\eta 0}^{(0)2}}\right)} \exp\left(-\frac{1}{8\varepsilon\sigma_{\eta_0}^{(0)2}}\right)\exp\left(-\frac{1}{2\sigma_{\eta_0}^{(0)2}}\left(B^2+\frac{\varepsilon}{2}B^4\right)\right), \tag{36}$$

where ω_{t0} is first the natural frequency of the overall system. $\sigma_{\eta_0}^{(0)2}$ represents the stationary variance of first modal displacement when å = 0. $\sigma_{\eta_0}^{(0)}$ is the root mean square of the displacement of the first modal response.

4. NUMERICAL RESULTS

Numerical calculation results of non-linear stochastic system are obtained under the random excitation while the system is rotating near the critical speed of the system. Non-linear characteristics of the random response are observed by the proposed analytical method and those are compared with the results by the numerical simulation.

4.1. MODEL FOR ANALYSIS

As an analysis model, a non-linear rotor system, which is shown in Figure 1, is considered. The rotor is considered as a uniform beam and the casing is also considered as a uniform beam approximately for the simplicity of calculation. The casing is constrained to a foundation at both ends of the casing. As a support, ball bearing is considered for aircraft engine turbine or power plant turbine. The properties of the rotor system are tabulated in Table 1.

The rotor is modelled by the 20 beam elements and the casing is also modelled by the 20 beam elements. The modal damping ratios of the system is given by $\zeta = 0{\cdot}05$. To check the computing accuracy of SSM, the natural frequency of the rotor system is calculated by using FEM, and the results are compared with those calculated by using SSM. The rotor has 84 d.o.f. because it has 21 nodes and there are 4 variables per mode. The casing also has 84 d.o.f. so that the total degrees of freedom are 168. Table 2 shows the natural frequencies of the rotor system calculated by FEM and SSM. In the analysis of SSM, the numbers of adopted modes in the rotor and casing are changed. The natural frequencies,

TABLE 1

Properties of the rotor system

Rotor, casing length (mm)	800
Rotor diameter (mm)	16
Casing diameter (mm)	50
Young's modulus (N/m^2)	$2{\cdot}1 \times 10^{11}$
Density of rotor and casing (kg/m^3)	$7{\cdot}81 \times 10^{3}$
Bearing coefficient (N/m)	$1{\cdot}0 \times 10^{6}$
Constrain coefficient (N/m)	$1{\cdot}0 \times 10^{10}$

TABLE 2

Natural frequency of rotor system (Hz)

Mode No.	FEM	SSM		
	168 (d.o.f.)	12/12	20/20	40/40
1	93·08	92·10	92·64	92·98
2	164·26	162·52	164·17	164·54

which are close to the ones by FEM, can be obtained by SSM when the numbers of adopted modes are increased.

When 20 modes are adopted in the rotor and casing, the error is less than 0·4%. To keep the accuracy of the lower natural frequencies, it is regarded that 20 modes of both components are sufficient to this analysis. Thus, 20 modes of each component are adapted to further response analysis for economic calculation. In the table (*A*/*B*), *A*, *B* stands for adopted mode number from the shaft/casing element.

The exciting force of the rotor caused by unbalance is assumed as $F_u(t) = F_0\Omega^2 \cos(\Omega t + \phi)$ in x direction, where F_0, ϕ are unbalance and phase angle, respectively. Rotating speed (Ω= 540 rad/s) is assumed to be near first critical speed (ω_{t1}= 583 rad/s) of the rotor system. The unbalance of the rotor system is located at the middle of the shaft with a value of 0·044 Kg m/(rad/s)^2.

4.2. RESPONSE OF THE NON-LINEAR SYSTEM

First, the linear response of the system against the random excitation is examined. The responses are computed by SSM with modal analysis procedure, which are shown in Figure 5. The statistic properties of the system response are examined. As can be noticed from Figure 5(a), the response includes the earthquake response of the system. Probability density of the system is shown in Figure 5(b) in accordance with relative amplitude of the system. Average value of response is 0·000186, which shows almost zero mean. These properties prove that the response is Gaussian stationary random process. In Figure 5(c), the power spectrum of the response is shown. It can be observed that a typical earthquake response power spectrum is obtained, which has a low-frequency component. Spectrum shows the fact that response is in a short period and predominant frequency is in a lower frequency where the peak frequency of the response is 28·27 rad/s. The simulated PSD shows well fitted with the analytical response of PSD $S_\eta(\Omega)$, which is obtained when $\varepsilon = 0$.

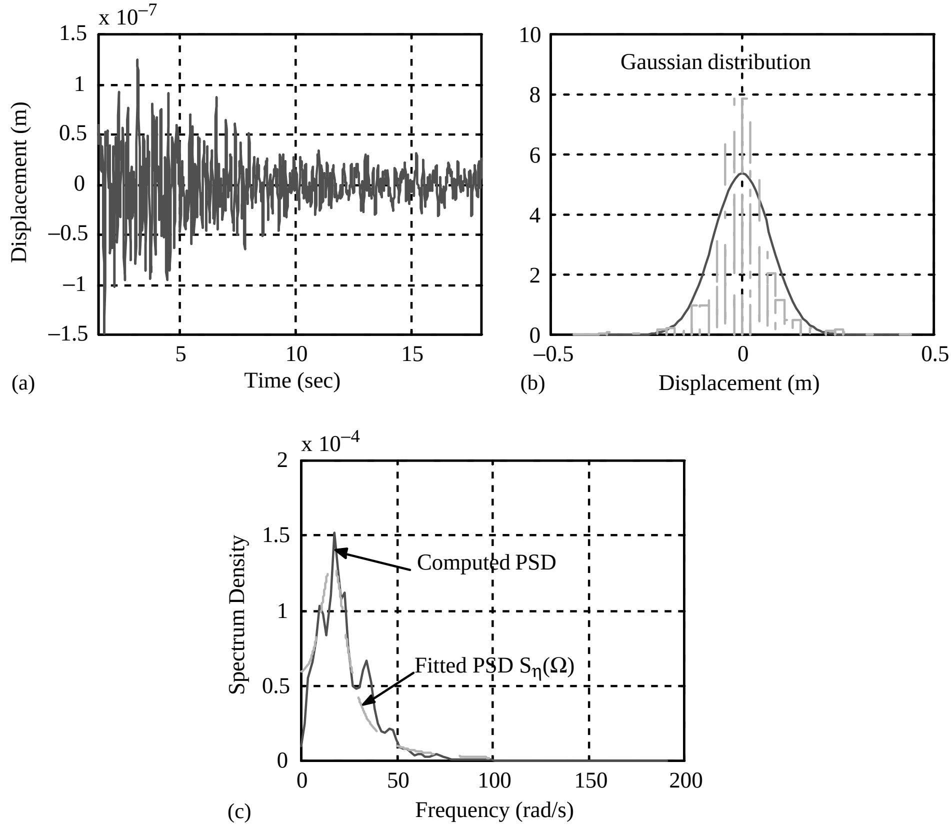

Figure 5. Linear response and its probabilistic properties. (a) Response of rotor, (b) Gaussian distribution, (c) power spectrum density.

The spectral density is related to the stationary variance $\sigma_{\eta_0}^{(0)2} = 0{\cdot}0059$, which is the same as the mean-square value of the linear response.

Here, the statistical properties of non-linear random vibration of the stochastic system versus the strong motion duration of excitation are investigated. Correlation, spectrum density and its variances, which are important to reliability analysis, are considered. Those properties of non-linear responses are calculated according to the procedure of non-linear random vibration analysis, which is computed to the perturbation first order. Spectrum of harmonic excitation and earthquake excitation are obtained and applied to the system. If the dual inputs are uncorrelated, then the cross-spectral density function of earthquake and unbalance excitation terms in the equation of spectral density become zero. Then, the autocorrelation and the spectral density of the non-linear response are obtained theoretically. It is regarded that the non-linear response depends on the size of the perturbation parameter, which shows nonlinear characteristics. To this end, two kinds of analysis are carried out, i.e., the perturbation parameters are set to $\varepsilon = 0{\cdot}2$, $0{\cdot}5$. In Figure 6, the PSD of the non-linear responses of the system at the middle of shaft and casing is shown with various non-linear parameters. The response of SSM is calculated by taking 20 modes. When the non-linear parameter is $\varepsilon = 0{\cdot}0$, the PSD is equal to a linear response. The each PSD of the system, such as rotor and casing, shows a characteristic of earthquake random process. Investigation of the PSD reveals that the PSD is smaller when

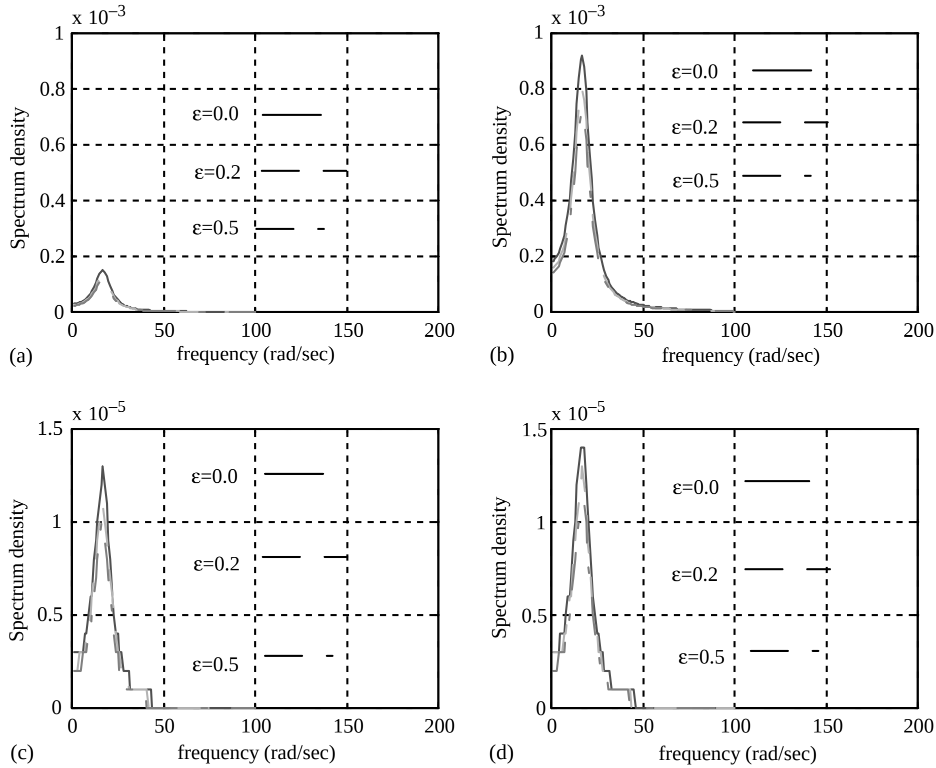

Figure 6. Comparing PSD of the responses. (a) Spectrum of rotor response (L), (b) spectrum of rotor response (C), (c) spectrum of casing response (L), (d) spectrum of casing response (C).

the perturbation parameter becomes bigger, which shows the non-linear characteristic of response. In the figure, (*L*), (*C*) stand for the left end nodal point and center of model.

Variance of the non-linear response, which is important value of reliability analysis in stochastic system, is evaluated from the non-linear response. Figure 7(a) shows the variance of the non-linear response at the middle of rotor by changing its numbers of adopting modes 20, 40. The calculated variances are investigated for various values of non-linear parameter. To prove the computing efficiency, those values are compared with the results of simulation, which is obtained with same calculation condition. The response of simulation is calculated by direct integration of the equation of motion against the excitation where the overall equation of motion of the system has 168 d.o.f. Investigation of the variance reveals that the value shows a decrease with ε in the spread of displacement about equilibrium point when $\varepsilon = 0$. This is consistent with our intuition, which suggests that stiffer systems exhibit smaller displacements, and with the observation that the system stiffness increases with non-linear parameter. This result also reveals that the variance displacement of a hardening spring non-linear system is always less than that for the corresponding linear system.

Figure 7(b) shows the variance of the non-linear response at the middle of rotor by using its numbers of adopting mode 20. The calculated variances are investigated for various values of non-linear parameter against maximum values of exciting acceleration. To prove the non-linear effect, those values are compared with the results of linear one, which is obtained by same calculation condition. It is observed that according to the input level,

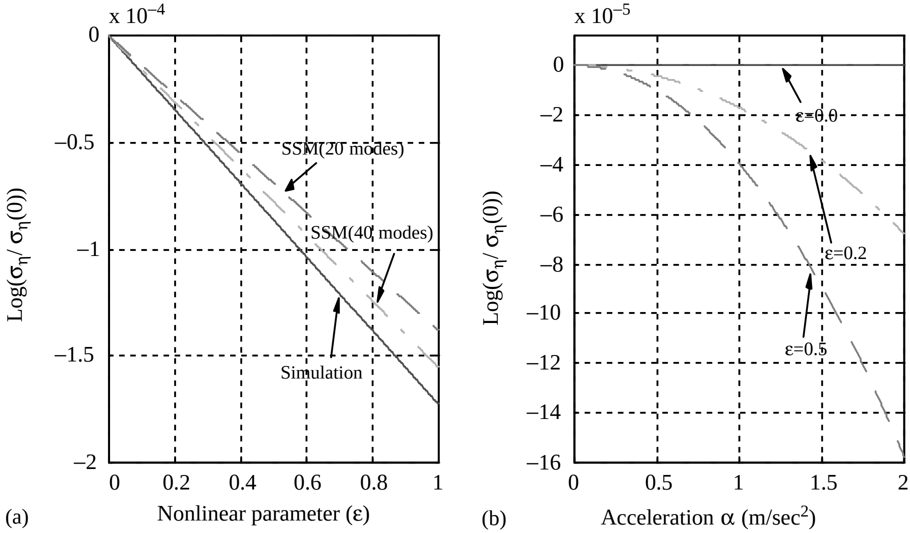

Figure 7. Displacement variance of the system. (a) Variance with non-linear parameter ε, (b) variance with acceleration α.

variance of the responses become large. When the non-linear parameters become large, then the response changes a lot.

Next, the computing time is compared to analyze the non-linear random vibration versus the strong motion duration (= 3–18 s) of excitation. The variances of response for the values to be compared are used, as shown in Figure 7(a). The computer used in this analysis is a *LOGIX, IBM* personal computer. In the case of the numerical simulation, it takes 20 min 45 s. But for the proposed method; it takes 5 min 28 s, 7 min 30 s to obtain the result, when the number of adopting modes are 20, 40 respectively. As a result, it can be shown that a drastic reduction in calculation time can be achieved, keeping its computing accuracy. This is an important factor in the analysis of structural dynamics against random excitation with a large number of degrees of freedom.

4.3. FAILURE POSSIBILITY

Reliability of the system can be obtained by supposing that the failure of the system occurs when the system exceeds the limit amplitude. To consider this, the limit of response (B) at rotor is regarded, which is constrained by casing to prevent the damage of bearing. This condition can be used to define a safe set. In this case, performance of the system depends primarily on crossing characteristics of safe set of response. Thus, the concept for mean crossing rate of safe set is applied to demonstrate the reliability of the system, as shown in Figure 8.

Figure 8 shows mean upcrossing rates of stationary non-linear responses for several values of non-linear parameter. The mean upcrossing rate decreases with threshold B due to a rapid reduction of probability density of the process reaching B. The mean upcrossing rate decreases more when non-linear parameters become large. These rates of the non-linear response can be used to approximate and bound the probability that the response or the restoring force does not exceed critical values of system. The reliability coincides with the mean upcrossing rate that the response belongs to the safe set. Also, the probability of failure can be obtained by the complement of obtained reliability.

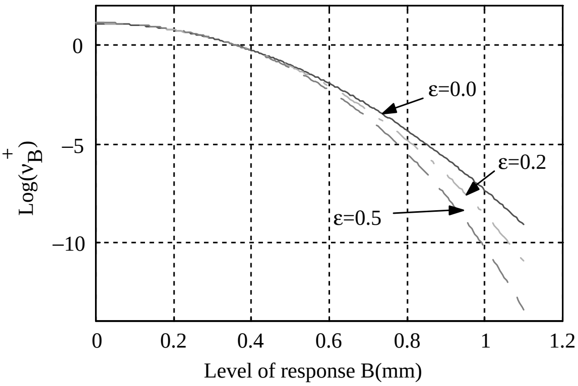

Figure 8. Mean *B*-upcrossing rate of response.

5. CONCLUSIONS

In this paper, the random vibration analysis method of a non-linear stochastic system was theoretically formulated when the actual random excitation is regarded as a Gaussian stationary random process. The formulation is concerned with reducing the number of degrees of freedom for each component by modal substitution. All of the components are then assembled together and the random response of the overall system is analyzed statistically against earthquake excitation. It is shown that non-linear random responses could be efficiently calculated according to the selected number of vibration modes. Several statistical properties of the random responses that are of interest in non-linear random vibration applications are reviewed. The variance value of the non-linear random response is obtained economically, which is important in evaluating the reliability of the system. The results reported herein provide a better understanding of the non-linear random vibration. Moreover, it is believed that those properties of the results can be utilized in the dynamic design of the non-linear stochastic system.

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